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A critical analysis of some fundamental differences in gauge approaches to gravitation

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Abstract. Static spherically symmetric solutions of a particular model of gravitation, including quadratic curvature and torsion terms, are discussed and compared with those found earlier in a model described by a different action. Considerable emphasis is placed on a Lorentz group covariant formulation of the models and the role of local diffeomorphisms of space–time carefully distinguished from the structure group for gravitation. We argue that the 'modified Poincaré gauge' approach of Hehl and co-workers mixes these concepts and that their model for gravity is dynamically specified by a locally Lorentz invariant action. Furthermore we demonstrate the logical distinction between a space–time transformation generated by symmetries of particular solutions to a model and gauge covariant symmetries that apply to all solutions.

1. Introduction

Since gauge theories seem important for the description of fundamental interactions it appears natural to exploit any gauge structure present in theories of gravity. Different authors, however, adopt different criteria in order to determine what properties a theory should possess in order for it to qualify as a gauge theory and there appears to be no consensus on what the gauge group should be. In a series of articles (Benn *et al* 1980a, b, 1981, Dereli and Tucker 1980, 1981a, b) we have formulated gravitational interactions in terms of an SL(2, C) structure group. Our approach entails the use of an SL(2, C) Lie algebra valued gauge field, gauging the covering of the Lorentz group that acts on local orthonormal frames of space–time and the assiduous use of intrinsic tensorial concepts. In this paper we show that modern differential geometric constructions, such as the linear frame bundle, are ideally suited to keep in focus a number of distinct concepts that can be easily blurred if one works entirely with tensor components on the space–time manifold itself. We explain why, and in what sense, gravitational interactions as we currently understand them should be formulated in terms of the local Lorentz gauge group rather than the Poincaré group or its modifications. This gauge structure is carefully distinguished from the coordinate invariance that all physical theories, not only those describing gravity, should exhibit.

Very soon after Einstein's announcement of general relativity, Cartan (1922) pointed out an important mathematical generalisation: the inclusion of space–time

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torsion. Since then many investigators, including Einstein himself, have sought to clarify the physical role of this particular generalisation. After the discovery of intrinsic spin it appeared even more natural and few would doubt the economy of describing gravitation in terms of independent metric and connection structures in the presence of self-gravitating fermions.

A model of gravity that involves quadratic combinations of curvature and/or torsion forms in the action (see e.g. Thirring 1978) has also attracted considerable attention in the literature, particularly when the torsion is not assumed to be *a priori* zero. In view of the dilemmas posed by the singularity theorems of Einsteinian gravity it may be worthwhile to study more closely these classical models.

In a recent paper by Benn *et al* (1981) we have found exact solutions to such a model. Since the model contains quadratic and linear curvature terms in the action we will refer to it as an Einstein–Cartan–Yang model although our modification includes a cosmological term. Formulated as an SL(2, C) gauge theory, the model yielded solutions under a simple double duality hypothesis that involved the curvature 2-form. We showed that such vacuum solutions, both with and without torsion, do not fall into the vacuum Einstein class. We learnt subsequently that similar solutions, involving torsion in the absence of matter, had been discovered independently by Baekler *et al* (1980) using a similar double duality approach. However in that case their model appears very different, involving in addition a quadratic torsion term and no explicit cosmological term in the action. At a more fundamental level we are instructed by these authors to regard their formulation as a 'modified Poincaré' gauge theory of gravity (Hehl 1979, 1980). In this paper we investigate their field equations to see if they are also satisfied by our solution to the Einstein–Cartan–Yang system. We are also prompted to discover the intrinsic nature of the 'modified Poincaré' gauge approach and compare it with our formulation of their model.

In § 2 we take the action of Baekler *et al* (1980) and reformulate it in an SL(2, C) gauge invariant manner and derive the coupled set of SL(2, C) gauge covariant field equations. Since the solutions under discussion are static and spherically symmetric we derive the form of the most general static spherically symmetric torsion using techniques based on Lie differentiation with respect to vector fields on space–time that generate SO(3). Using our double duality condition we then show explicitly that both sets of field equations are satisfied by our solution. Since this solution has a torsion that is not SL(2, C) gauge equivalent to those presented by Baekler *et al* (1980) we must conclude that this particular action (like the Einstein–Cartan–Yang model in Benn *et al* (1981)) does not admit unique static spherically symmetric solutions with non-vanishing torsion. Throughout this section we make explicit use of the SL(2, C) covariance to choose a gauge that is very suited to spherically symmetric field configurations.

In § 3 we describe briefly the concept of the linear frame bundle over space–time and show that the infinitesimal 'modified Poincaré' transformations can be understood as local Lorentz transformations combined with a Lie derivative with respect to an arbitrary vector field on M. In view of this we are persuaded to argue that the latter should not be regarded as defining part of a structure group associated with a dynamic description of gravitation. Rather, we argue that the behaviour of particular geometries and other field configurations (arising from a locally Lorentz invariant theory) under the action of particular Lie derivatives serves to indicate their symmetry under diffeomorphisms on M. In § 4 this viewpoint is contrasted briefly with the closely allied concept of a Cartan connection (Kobayashi and Nomizu 1969) on the affine frame bundle.

Our arguments are summarised in the conclusion.

2.

A manifold M, such as space–time, is defined in terms of its topology, its atlas of charts and the smoothness properties of maps between them. The primitive requirement of the coordinate independence of physical laws can be accommodated by formulating theories in terms of tensors on M. In the first part of this section we work entirely with real valued differential forms on M and explore solutions in a local chart of M. The latter subset of all tensors plays a prominent role since the theories under discussion are generated from an action integral on M. Differential 4-forms can be integrated over a 4-chain in a coordinate independent manner.

The action of Baekler *et al* (1980) can be rewritten in terms of the 4-form

$$\Lambda = kR^{ab} \wedge * R_{ab} + \tfrac{1}{2}T^a \wedge * T_a - \tfrac{1}{2}\alpha \wedge * \alpha \qquad a, b = 0, 1, 2, 3 \tag{1}$$

and the usual summation convention is operative with latin indices here raised and lowered with the matrix $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. k is a real constant and for convenience we shall use the abbreviations $\alpha = i_a T^a$, $\sigma^a = \alpha \wedge e^a$ where the interior derivation is with respect to the dual frame

$$i_a e^b = \delta^b_a. \tag{2}$$

(Those unfamiliar with i_a will find its general definition in § 3.) The space–time metric g, torsion 2-forms T^a and curvature 2-forms $R^a{}_b$ are given by

$$g = -e^0 \otimes e^0 + \sum_{k=1}^{3} e^k \otimes e^k \tag{3}$$

$$T^a = \mathrm{d}e^a + \psi^a{}_b \wedge e^b \tag{4}$$

$$R^a{}_c = \mathrm{d}\omega^a{}_c + \omega^a{}_b \wedge \omega^b{}_c \tag{5}$$

in terms of the metric compatible connection 1-forms $\omega_{ab} = -\omega_{ba}$ and $*$ denotes the Hodge dual map.

The real form field equations that follow from this action by varying e^a and $\omega^a{}_b$ are respectively

$$\begin{aligned} D * (T_a + \sigma_a) = {} & \tfrac{1}{2}\{i_a T^b \wedge * T_b - T^a \wedge i_a * T_b\} + \tfrac{1}{2}\{i_a\alpha \wedge * \alpha + \alpha \wedge i_a * \alpha\} + i_a T^b \wedge * \sigma_b \\ & - \sigma^b \wedge i_a * T_b - \alpha \wedge * T_a + k[i_a R^{bc} \wedge * R_{bc} - R^{bc} \wedge i_a * R_{bc}] \end{aligned} \tag{6}$$

$$4kD * R^{ab} + T^c \wedge * (e^a \wedge e^b \wedge e_c) = e^a \wedge e^b \wedge i_c * T^c. \tag{7}$$

In these expressions D denotes the appropriate $\mathrm{SL}(2, C)$ exterior covariant derivative.

At this point, to compare with the language used in our earlier paper (Benn *et al* 1981), we recast these equations into a complex quaternionic form. This is most readily accomplished by making variations in the action written in terms of the anti-Hermitian (Tucker 1981) quaternionic 2-form $T = \mathrm{i}T^0 + \Sigma_{k=1}^{3} T^k \hat{e}_k$ and the complex quaternionic 2-form $\hat{R} = R^k \hat{e}_k$. The $\hat{e}_k$ are three elements that generate the quaternion algebra.

$$\Lambda = -8k \,\mathrm{Re}\, S[\hat{R} \wedge * \hat{R}] + \tfrac{1}{2}S(T \wedge * \bar{T}) - \tfrac{1}{2}\alpha \wedge * \alpha. \tag{8}$$

The quaternionic field equations corresponding to (6) and (7) become

$$\begin{aligned} D * (\bar{T} + \bar{\sigma}) & + \tfrac{1}{2}\{u_X T^a \wedge * T_a - T^a \wedge u_X * T_a\} + \tfrac{1}{2}\{u_X \alpha \wedge * \alpha + \alpha \wedge u_X * \alpha\} \\ & - \alpha \wedge * \bar{T} + u_X T^a \wedge * \sigma_a - \sigma^a \wedge u_X * T_a + 8k\tau = 0 \end{aligned} \tag{9}$$

$$16\, kD * \hat{R} + \mathrm{i}D(e \wedge \bar{e}) = (e \wedge \bar{e}) \wedge \beta. \tag{10}$$

We have introduced

$$e = \mathrm{i}e^0 + \sum_{k=1}^{3} e^k \hat{e}_k \tag{11}$$

$$u_X = \mathrm{i}i_0 + \sum_{k=1}^{3} i_k \hat{e}_k \tag{12}$$

$$\alpha = -S(u_X T)^* \tag{13}$$

$$\beta = -S(u_X * T)^* \tag{14}$$

$$\tau = \mathrm{i}\tau_0 + \sum_{k=1}^{3} \tau_k \hat{e}_k = 2\mathscr{A}\left\{ \sum_{k=1}^{3} u_X R^k \wedge * R^k + u_X S(\hat{R} \wedge * \hat{R}) \right\}. \tag{15}$$

In terms of real forms

$$\tau_a = -\tfrac{1}{8}[i_a R^{bc} \wedge * R_{bc} - R^{bc} \wedge i_a * R_{bc}] \tag{16}$$

are the four stress 3-forms associated with the first term in the action. (NB a right superscript * denotes complex conjugation.)

Equations (7) or (10) are solved immediately if one adopts the conditions

$$\beta = 0 \tag{17}$$

$$R_{ab} = -\tfrac{1}{2}\varepsilon_{ab}{}^{cd} * R_{cd} - \tfrac{1}{4}\lambda e_a \wedge e_b \tag{18}$$

provided the constants satisfy $k\lambda = 1$. The second condition can be recognised as the modified double duality used by us in Benn *et al* (1981)

$$\hat{R} = \mathrm{i} * \hat{R} + \tfrac{1}{16}\lambda e \wedge \bar{e}. \tag{19}$$

To see that (10) is solved one simply applies an exterior covariant derivative to this equation and uses the Bianchi identity $D\hat{R} = 0$.

To solve the frame variation equation is more involved and one must exploit static spherical symmetry of a geometry containing torsion.

In order to make this constraint precise we employ the notion of a Lie derivative $\mathscr{L}_X$ with respect to a vector field X on M. We first establish coordinate functions (t, r, θ, ϕ) which define a coordinate patch on M by giving them values in R^4 satisfying

$$0 \leq t < \infty \qquad 0 \leq r < \infty \qquad 0 \leq \theta < \pi \qquad 0 \leq \phi < 2\pi. \tag{20}$$

The metric is static and spherically symmetric if the four vector fields

$$\begin{aligned} &Y_0 = \frac{\partial}{\partial t} \qquad Y_1 = -\sin\phi \frac{\partial}{\partial \theta} - \cos\phi \cot\theta \frac{\partial}{\partial \phi} \\ &Y_2 = \cos\phi \frac{\partial}{\partial \theta} - \sin\phi \cot\theta \frac{\partial}{\partial \phi} \qquad Y_3 = \frac{\partial}{\partial \phi} \end{aligned} \tag{21}$$

define a Lie derivative that annihilates g

$$\mathscr{L}_{Y_i} g = 0 \qquad i = 0, 1, 2, 3. \tag{22}$$

The vector fields Y_1, Y_2, Y_3 generate the Lie algebra of SO(3) which acts as an infinitesimal transformation group on M. In terms of these coordinates one readily verifies that the four coframe forms

$$e^0 = h_0(r)\,\mathrm{d}t \qquad e^1 = h_1(r)\,\mathrm{d}r \qquad e^2 = h_2(r)\,\mathrm{d}\theta \qquad e^3 = h_2(r)\sin\theta\,\mathrm{d}\phi \tag{23}$$

provide an orthonormal basis for g satisfying (22) where h_0, h_1, h_2 are arbitrary real functions of r.

A dual frame $\{X_i\}$ defined by $e^i(X_j) = \delta^i_j$ is given by

$$X_0 = \frac{1}{h_0}\frac{\partial}{\partial t} \qquad X_1 = \frac{1}{h_1}\frac{\partial}{\partial r} \qquad X_2 = \frac{1}{h_2}\frac{\partial}{\partial\theta} \qquad X_3 = \frac{1}{h_2\sin\theta}\frac{\partial}{\partial\phi} \tag{24}$$

which satisfy

$$\begin{aligned} &\mathscr{L}_{Y_1}X_2 = -\cos\phi\,\operatorname{cosec}\theta\,X_3 \qquad &&\mathscr{L}_{Y_2}X_2 = -\sin\phi\,\operatorname{cosec}\theta\,X_3 \\ &\mathscr{L}_{Y_1}X_3 = \operatorname{cosec}\theta\cos\phi\,X_2 &&\mathscr{L}_{Y_2}X_3 = \sin\phi\,\operatorname{cosec}\theta\,X_2 \end{aligned} \tag{25}$$

with all other Lie derivatives zero. The complete geometry is said to be static and spherically symmetric if the Lorentz gauge invariant torsion (1, 2) tensor

$$\mathbb{T} = T^a \otimes X_a \tag{26}$$

obeys

$$\mathscr{L}_{Y_i}\mathbb{T} = 0 \qquad i = 0, 1, 2, 3. \tag{27}$$

One readily computes that this implies

$$\begin{aligned} &T^0 = Ae^0\wedge e^1 + Be^2\wedge e^3 \qquad T^1 = Ce^0\wedge e^1 + De^2\wedge e^3 \\ &T^2 = Ee^0\wedge e^2 + Fe^0\wedge e^3 + Ge^1\wedge e^2 + He^1\wedge e^3 \\ &T^3 = Ee^0\wedge e^3 - Fe^0\wedge e^2 + Ge^1\wedge e^3 - He^1\wedge e^2 \end{aligned} \tag{28}$$

where A to H are real functions of r alone. The coordinate reflection $(t, r, \theta, \phi)\to(t, r, \pi-\theta, \pi+\phi)$ induces the frame reflection $(e^0, e^1, e^2, e^3)\to(e^0, e^1, -e^2, e^3)$ which may be used to partition the eight functions into two sets depending on the nature of the behaviour of their associated basis 2-forms under this transformation.

To simplify subsequent calculations we shall now exploit the Lorentz gauge freedom to select a polar gauge that is particularly suited to the spherical symmetry. We seek a gauge transformation with the property

$$e \to \tilde{e} = \mathrm{i}h_0\,\mathrm{d}t + h_1N\,\mathrm{d}r + h_2\,\mathrm{d}N = QeQ^\dagger \tag{29}$$

where $N(\theta, \phi) = \hat{e}_1\cos\theta + \hat{e}_2\exp(-\hat{e}_1\phi)\sin\theta$ is a real unit q vector $(N\bar{N} = 1)$ and $Q \in \mathrm{SL}(2, C)$. This requires that the local gauge rotation Q obey

$$QQ^\dagger = 1 \qquad Q\hat{e}_1\bar{Q} = N \qquad Q\hat{e}_2\bar{Q} = UN \qquad Q\hat{e}_3\bar{Q} = U \tag{30}$$

where $U = \hat{e}_3\exp(-\hat{e}_1\phi)$. By explicit calculation

$$Q = \exp(\tfrac{1}{2}\hat{e}_1\phi)\exp(\tfrac{1}{2}\hat{e}_3\theta) \tag{31}$$

and corresponds to an SO(3) rotation. Thus we can compute the torsion 2-form in the new gauge

$$\begin{aligned} \tilde{T} = QTQ^\dagger = {}&\mathrm{i}A_1^{(+)}\,\mathrm{d}r\wedge\mathrm{d}t + A_2^{(-)}\,\mathrm{d}r\wedge N\,\mathrm{d}N + A_3^{(-)}\,\mathrm{d}t\wedge N\,\mathrm{d}N \\ &+ A_4^{(-)}\,\mathrm{d}N\wedge\mathrm{d}N + \mathrm{i}B_1^{(-)}N\,\mathrm{d}N\wedge\mathrm{d}N + B_2^{(+)}\,\mathrm{d}r\wedge\mathrm{d}N \\ &+ B_3^{(+)}\,\mathrm{d}t\wedge\mathrm{d}N + B_4^{(+)}\,\mathrm{d}r\wedge\mathrm{d}t\,N \end{aligned} \tag{32}$$

and the superscripts on the components indicate the behaviour under the frame reflection above. These components are related to the functions in (26) by

$$A_1^{(+)} = A \qquad A_2^{(-)} = -rH/h_0 \qquad A_3^{(-)} = -rh_0F \qquad A_4^{(-)} = \tfrac{1}{2}r^2D$$
$$B_1^{(-)} = -\tfrac{1}{2}r^2B \qquad B_2^{(+)} = rG/h_0 \qquad B_3^{(+)} = rh_0E \qquad B_4^{(+)} = -C. \tag{33}$$

It is also very convenient to rotate the interior operator into this gauge. We find

$$u_X \to \tilde{u}_X = \frac{\mathrm{i}}{h_0} i_t + \frac{N}{h_1} i_r + \frac{2}{h_2}\hat{i}_N = \bar{Q}^\dagger u_X \bar{Q} \tag{34}$$

where

$$i_t = i_{\partial/\partial t} \qquad i_r = i_{\partial/\partial r}$$

and $\hat{i}_N$ obeys

$$\hat{i}_N(\mathrm{d}N) = -1 \qquad \hat{i}_N(N\,\mathrm{d}N) = N \qquad \hat{i}_N(\mathrm{d}N\,\mathrm{d}N) = -\mathrm{d}N$$
$$\hat{i}_N(N\,\mathrm{d}N\,\mathrm{d}N) = N\,\mathrm{d}N.$$

Since the single q vector N now replaces the θ, ϕ dependence we give the duality relations in the polar gauge

$$* 1 = -\tfrac{1}{2}h_2^2h_0h_1\,\mathrm{d}r \wedge \mathrm{d}t \wedge N\,\mathrm{d}N\,\mathrm{d}N = e^1 \wedge e^2 \wedge e^3 \wedge e^0$$
$$* \mathrm{d}r = -\tfrac{1}{2}h_2^2(h_0/h_1)N\,\mathrm{d}N \wedge \mathrm{d}N \wedge \mathrm{d}t$$
$$* \mathrm{d}t = -\tfrac{1}{2}h_2^2(h_1/h_0)N\,\mathrm{d}N \wedge \mathrm{d}N \wedge \mathrm{d}r \qquad * \mathrm{d}N = h_0h_1N\,\mathrm{d}N \wedge \mathrm{d}t \wedge \mathrm{d}r$$
$$* (\mathrm{d}N \wedge \mathrm{d}N) = \frac{2N}{h_2^2}h_0h_1\,\mathrm{d}r \wedge \mathrm{d}t \qquad * (\mathrm{d}r \wedge \mathrm{d}t) = \frac{h_2^2}{2h_0h_1}N\,\mathrm{d}N \wedge \mathrm{d}N \tag{35}$$
$$* (\mathrm{d}r \wedge \mathrm{d}N) = -(h_0/h_1)N\,\mathrm{d}N \wedge \mathrm{d}t \qquad * (\mathrm{d}t \wedge \mathrm{d}N) = -(h_1/h_0)N\,\mathrm{d}N \wedge \mathrm{d}r.$$

During subsequent calculations we repeatedly use the fact that $N\,\mathrm{d}N \wedge \mathrm{d}N$ is a real q scalar while N, $N\,\mathrm{d}N$, $\mathrm{d}N$ and $\mathrm{d}N \wedge \mathrm{d}N$ are all real q vectors. We shall leave the over tilde understood in the polar gauge.

Returning to the solution of (9) we recall from Benn *et al* (1981) that the solutions therein corresponding to the action

$$\Lambda = -\mathrm{Re}\,S\{2R \wedge * R + \mathrm{i}(2/\lambda)\hat{R} \wedge e \wedge \bar{e} + \mathrm{i}\tfrac{1}{4}\lambda^{-2}e \wedge \bar{e} \wedge e \wedge \bar{e}\} \tag{36}$$

are

$$g = -\left(1 + \frac{c}{r} + \frac{kr^2}{3}\right)\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}r \otimes \mathrm{d}r\left(1 + \frac{c}{r} + \frac{kr^2}{3}\right)^{-1} + r^2(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta\,\mathrm{d}\phi \otimes \mathrm{d}\phi) \tag{37}$$

$$T = -\mathrm{i}\left[2\left(\frac{k}{3} - \frac{1}{\lambda}\right)r - \frac{c}{r^2}\right]e^0 \wedge e^1\left(1 + \frac{c}{r} + \frac{kr^2}{3}\right)^{-1/2} \tag{38}$$

where λ, c and k are arbitrary real constants.

We consequently assume a solution to the field equation (9), (10) of the form

$$e = \mathrm{i}h_0\,\mathrm{d}t + N\,\mathrm{d}r/h_0 + r\,\mathrm{d}N \tag{39}$$

$$T = \frac{2\mathrm{i}f_0}{h_0}\mathrm{d}r \wedge \mathrm{d}t \tag{40}$$

where f_0 and h_0 are real functions of r. With the aid of the structure equation

$$T = \mathrm{d}e + 2\mathscr{A}(\hat{\omega} \wedge e) \tag{41}$$

the corresponding SL(2, C) connection is

$$\hat{\omega} = \mathrm{i}(p_0 + f_0)N\,\mathrm{d}t + p_1 N\,\mathrm{d}N \tag{42}$$

with

$$p_0 = -\tfrac{1}{2}h_0\partial_r h_0 \qquad p_1 = \tfrac{1}{2}(h_0 - 1).$$

We may now compute

$$\begin{aligned}\hat{R} &= \mathrm{d}\hat{\omega} + \hat{\omega} \wedge \hat{\omega} \\ &= \mathrm{i}\partial_r(p_0+f_0)N\,\mathrm{d}r \wedge \mathrm{d}t - \mathrm{i}h_0(p_0+f_0)\,\mathrm{d}t \wedge \mathrm{d}N \\ &\quad + \tfrac{1}{2}\partial_r h_0\,\mathrm{d}r \wedge N\,\mathrm{d}N + \tfrac{1}{4}(h_0^2 - 1)\,\mathrm{d}N \wedge \mathrm{d}N\end{aligned} \tag{43}$$

$$e \wedge \bar{e} = 2\mathrm{i}N\,\mathrm{d}r \wedge \mathrm{d}t - 2\mathrm{i}rh_0\,\mathrm{d}t \wedge \mathrm{d}N - (2r/h_0)\,\mathrm{d}r \wedge N\,\mathrm{d}N - r^2\,\mathrm{d}N \wedge \mathrm{d}N \tag{44}$$

$$\begin{aligned}\mathrm{i} * \hat{R} &= (\mathrm{i}/2r^2)(h_0^2 - 1)N\,\mathrm{d}r \wedge \mathrm{d}t - \tfrac{1}{2}\mathrm{i}h_0^2\partial_r h_0\,\mathrm{d}N \wedge \mathrm{d}N \\ &\quad + h_0^{-1}(p_0+f_0)\,\mathrm{d}r \wedge N\,\mathrm{d}N + \partial_r(p_0+f_0)\tfrac{1}{2}r^2\,\mathrm{d}N \wedge \mathrm{d}N.\end{aligned} \tag{45}$$

Substituting in the modified duality condition (19) gives the differential equations that must be satisfied

$$p_0 + f_0 + \tfrac{1}{8}r\lambda = \tfrac{1}{2}h_0\partial_r h_0 \tag{46}$$

$$\partial_r p_0 + \partial_r f_0 + \tfrac{1}{8}\lambda = (h_0^2 - 1)/2r^2 \tag{47}$$

with solutions

$$h_0 = (1 + A/r + Br^2)^{1/2} \tag{48}$$

$$f_0 = (B - \tfrac{1}{8}\lambda)r - A/2r^2 \tag{49}$$

where A and B are arbitrary integration constants. Thus the ansatz (39)–(40) solves the equation (10). It is now straightforward to calculate the terms that enter into (9) using the polar gauge

$$* T = (\mathrm{i}f_0/h_0)r^2 N\,\mathrm{d}N \wedge \mathrm{d}N \tag{50}$$

$$\alpha = -(2f_0/h_0^2)\,\mathrm{d}r \tag{51}$$

$$* \alpha = r^2 f_0\,\mathrm{d}t\,N\,\mathrm{d}N \wedge \mathrm{d}N \tag{52}$$

$$\sigma = -(2\mathrm{i}f_0/h_0)\,\mathrm{d}r \wedge \mathrm{d}t - (2f_0 r/h_0^2)\,\mathrm{d}r \wedge \mathrm{d}N \tag{53}$$

$$* \sigma = -\mathrm{i}(f_0 r^2/h_0)N\,\mathrm{d}N \wedge \mathrm{d}N + 2f_0 rN\,\mathrm{d}N \wedge \mathrm{d}t. \tag{54}$$

With the aid of u_X we find

$$\begin{aligned}&u_X T^a * T_a - T^a u_X * T_a + u_X\alpha \wedge * \alpha \\ &\quad + \alpha u_X * \alpha + 2u_X T^a \wedge * \sigma_a - 2\sigma^a \wedge u_X * T_a - 2\alpha \wedge * \bar{T} = 0.\end{aligned} \tag{55}$$

Now

$$\begin{aligned}D * (\bar{T} + \bar{\sigma}) &= \mathrm{d} * (\bar{T} + \bar{\sigma}) - 2\mathscr{A}(* (\bar{T} + \bar{\sigma}) \wedge \hat{\omega}) \\ &= [A/r^2 + 4r(B - \tfrac{1}{8}\lambda)]\,\mathrm{d}r \wedge \mathrm{d}t \wedge N\,\mathrm{d}N - 2rf_0h_0\,\mathrm{d}t \wedge \mathrm{d}N \wedge \mathrm{d}N\end{aligned} \tag{56}$$

and this term must be balanced by the remaining stress forms in (9)

$$\tau=\sum_{k=1}^{3} 2\mathscr{A}(u_X R^k \wedge * R^k)+\mathscr{A}(u_X S(\hat{R}\wedge * \hat{R})) \tag{57}$$

where

$$R^1=-S(\hat{R}N) \qquad R^2=-S(\hat{R}UN) \qquad R^3=-S(\hat{R}U).$$

Thus

$$\tau=\left(-\frac{\lambda A}{8r^2}+\frac{r\lambda}{8}(\tfrac{1}{8}\lambda-B)\right)\mathrm{d}r\wedge\mathrm{d}t\,N\wedge\mathrm{d}N+\left(\frac{\lambda r^2 h_0}{16}(\tfrac{1}{8}\lambda-B)-\frac{\lambda}{8r}Ah_0\right)\mathrm{d}t\wedge\mathrm{d}N\wedge\mathrm{d}N$$
$$+\mathrm{i}\frac{\lambda r^2}{16h_0}(\tfrac{1}{8}\lambda-B)\,\mathrm{d}r\wedge N\,\mathrm{d}N\wedge\mathrm{d}N. \tag{58}$$

Equation (9) is consequently solved if we set $B=\frac{1}{8}\lambda$. Thus by suitably adjusting the constants we find that our previous solution (37), (38)

$$g=-\left(1+\frac{A}{r}+\frac{r^2}{8k}\right)\mathrm{d}t\otimes\mathrm{d}t+\mathrm{d}r\otimes\mathrm{d}r\left(1+\frac{A}{r}+\frac{r^2}{8k}\right)^{-1}+r^2(\mathrm{d}\theta\otimes\mathrm{d}\theta+\sin^2\theta\,\mathrm{d}\phi\otimes\mathrm{d}\phi) \tag{59}$$

$$\begin{aligned} T^0&=-Ar^{-2}\left(1+\frac{A}{r}+\frac{r^2}{8k}\right)^{-1/2}\mathrm{d}r\wedge\mathrm{d}t \\ T^k&=0 \qquad k=1,2,3 \end{aligned} \tag{60}$$

also solves exactly the field equations based on the action (8). The most general expression for static, spherically symmetric torsion 2-form with positive reflection symmetry is

$$T=(\mathrm{i}2f_0/h_0)\,\mathrm{d}r\wedge\mathrm{d}t-2h_0f_1N\,\mathrm{d}r\wedge\mathrm{d}t+2h_0f_2\,\mathrm{d}t\wedge\mathrm{d}N-(2f_3/h_0)\,\mathrm{d}r\wedge\mathrm{d}N \tag{61}$$

where f_0, f_1, f_2, f_3 are real functions of r alone. The corresponding (q-vector valued) contortion 1-form

$$\hat{K}=\mathrm{i}f_0N\,\mathrm{d}t+\mathrm{i}f_1N\,\mathrm{d}r+\mathrm{i}f_2\,\mathrm{d}N+f_3N\,\mathrm{d}N \tag{62}$$

defines the torsion $T=2\mathscr{A}(\hat{K}\wedge e)$.

This ansatz, together with the expression (39) specifying the ansatz for the basis 1-forms, determines the curvature 2-form

$$\hat{R}=-\mathrm{i}(p_0+f_0)(1+2p_1+2f_3)\,\mathrm{d}t\wedge\mathrm{d}N+\mathrm{i}(\partial_r p_0+\partial_r f_0)N\,\mathrm{d}r\wedge\mathrm{d}t$$
$$+[(1+p_1+f_3)(p_1+f_3)-f_2^2]\,\mathrm{d}N\wedge\mathrm{d}N-2(p_0+f_0)f_2\,\mathrm{d}t\wedge N\,\mathrm{d}N$$
$$-\mathrm{i}[f_1(1+2p_1+2f_3)-\partial_r f_2]\,\mathrm{d}r\wedge\mathrm{d}N+(\partial_r p_1+\partial_r f_3-2f_1f_2)\,\mathrm{d}r\wedge N\,\mathrm{d}N. \tag{63}$$

When this expression is substituted in the modified double duality condition $\hat{R}=\mathrm{i}*\hat{R}-\frac{1}{16}\lambda e\wedge\bar{e}$, the following set of ordinary differential equations is obtained:

$$\partial_r p_0+\partial_r f_0+\tfrac{1}{8}\lambda=\tfrac{1}{2}r^{-2}[(h_0+2f_3)^2-4f_2^2-1] \tag{64}$$

$$(p_0+f_0)(1+2f_3/h_0)+\tfrac{1}{8}\lambda r=\tfrac{1}{2}h_0(\partial_r h_0+2\partial_r f_3-4f_1f_2) \tag{65}$$

$$2(p_0+f_0)f_2=h_0^2[\partial_r f_2-f_1(h_0+2f_3)]. \tag{66}$$

Besides the solution given above (determined by the assumption $f_0 \neq 0, f_1 = f_2 = f_3 = 0$) there are two other distinct solutions. They are specified by the metric function

$$h_0 = (1 + A/r + Br^2)^{1/2} \tag{67}$$

together with the functions

Case I $\quad f_0 = -A/4r^2 \qquad f_1 = A/4r^2h_0^2 \qquad f_2 = -A/4rh_0 \qquad f_3 = -A/4rh_0$ (68)

and

Case II $\quad f_0 = -A/4r^2 \qquad f_1 = -A/4r^2h_0^2 \qquad f_2 = A/4rh_0 \qquad f_3 = A/4rh_0$ (69)

respectively. These latter solutions correspond in this gauge to those discovered by Baekler *et al* (1980) and were shown to satisfy the frame variation equations (6) as well. It should be noted here that the solution (59)–(60) presented previously in detail meets all the criteria Baekler *et al* require their solutions to satisfy and was in fact overlooked in their paper. We also find it interesting to note that all these three cases constitute distinct solutions to the field equations of the Einstein–Cartan–Yang theory with a cosmological term (Benn *et al* 1981). We have not investigated solutions to either model corresponding to the most general spherically symmetric static torsion.

3.

In this section we consider the nature of the 'translations' that Baekler *et al* (1980) describe as being gauged. Our use of tensors on M in the previous calculation explicitly accommodates the notion of coordinate independence regarded as relabelling conventions for points on M. However the notion of general coordinate transformations has also been identified with the concept of local diffeomorphisms on M which describe the motion of its points under a smooth 1–1 mapping. These concepts must be clearly distinguished. In the former case $h: R^4 \to R^4$ is a diffeomorphism between local charts, each of which gives a definite coordinate label to the same point of M. In the latter case $f: M \to M$ maps the manifold smoothly onto itself. Since this map is specified by its representation in local charts (which may coincide) there is room for ambiguity unless it is clear from the context which case is being considered. Baekler *et al* (1980) clearly have active diffeomorphisms $M \to M$ in mind which they refer to as translations.

In Minkowski space there exists a privileged global coordinate system $(t, x^k) \in R^4$ in which the space–time metric takes the form

$$g = -\mathrm{d}t \otimes \mathrm{d}t + \sum_{k=1}^{3} \mathrm{d}x^k \otimes \mathrm{d}x^k. \tag{70}$$

The Poincaré group, realised as a transformation group on M, is a symmetry of g. It consists of ten infinitesimal generators $X_i \in TM$ with the properties

$$\mathscr{L}_{X_i} g = 0. \tag{71}$$

The four pure translation generators take a form

$$X_i = \varepsilon_0 \partial/\partial t \tag{72}$$

$$X_k = \varepsilon_k \partial/\partial x^k \qquad k = 1, 2, 3 \text{ no sum} \tag{73}$$

where in these coordinates the $\varepsilon_0, \varepsilon_k$ are constants. Such vector fields generate a diffeomorphism of Minkowski space onto itself given in a global chart by

$$t \to t + \varepsilon_0 \qquad x_k \to x_k + \varepsilon_k \tag{74}$$

which by (71) is an isometry. Under a coordinate relabelling (71) is still valid of course, but the vector fields no longer need to have constant functions for their components in the new coordinate basis. Vector fields with constant components in arbitrary coordinate bases do not generate isometries in Minkowski space. What then is the role of a set of arbitrary vector fields in a space–time with an arbitrary metric?

The notion of a gauge transformation means different things to different authors. Most physicists use of the gauge principle stems from the belief in the local independence of certain transformations. From a pragmatic viewpoint it is related to the existence of covariant (or invariant) expressions that compare objects at distinct points in space–time. This has led to the construction of intrinsic derivatives in terms of maps that provide meaningful ways of comparing tensors at different points.

Given an arbitrary one-parameter group of diffeomorphisms $\phi_t : M \to M$ one can use its generator X (where X_p is tangent to an integral curve through $p \in M$) to perform Lie differentiation on an arbitrary tensor W

$$\mathscr{L}_X W = \lim_{t\to 0} (W - \tilde{\phi}_t W)/t \tag{75}$$

where $\tilde{\phi}_t W$ is the action on W induced by ϕ_t. This gives information on the behaviour of different types of tensor induced by such a diffeomorphism. It is a fundamental method of differentiation and requires the existence only of a differentiable manifold and a local vector field. In particular it exists in the absence of any geometric structure on M. It is for this reason that we would argue that its role be distinguished from higher-level methods of differentiation that require extra structure such as a metric and linear connection. In the context of space–time it is the latter constructs that we identify with the gravitational field. Although we do not deny the great utility of the Lie derivative in describing the evolution of the gravitational field, the basic geometry of M is to be found in terms of structures on the linear frame bundle (Kobayashi and Nomizu 1969, Schmidt 1974, Trautman 1980).

On any manifold a frame u at a point $p \in M$ is an ordered basis of tangent vectors $\{X_i\}$ associated with that point. The set of all frames will be denoted here by LM and Kobayashi and Nomizu (1969) indicate how this space may be endowed with the structure of a principal fibre bundle over M with $\mathrm{Gl}(4, R)$ as structure group if M has four dimensions. If $\{x^\mu\}$ coordinates points in a region $U \subset M$ then one may take the $4 + 16$ numbers $\{x^\mu, e^\mu_a\}$ as the coordinates of a region $\pi^{-1}(U) \subset LM$ where all indices run from 0, 1, 2, 3 and using the summation convention

$$X_i = e^\mu_i\, \partial/\partial x^\mu \tag{76}$$

relates the natural coordinate vectors $\partial/\partial x^\mu$ to an arbitrary frame X_i by the matrix elements of $\mathrm{Gl}(4, R)$. Under $A \in \mathrm{Gl}(4, R)$

$$(x^\mu, e^\mu_i) \to (x^\mu, e^\mu_k A^k_i) \tag{77}$$

gives the group action on LM in these coordinates. A passive M coordinate transformation that relabels $p \in M$

$$x^k \to x^{k'}(x) \tag{78}$$

can induce a particular $\mathrm{Gl}(4, R)$ transformation with matrix

$$A_i^k = \partial x'^k / \partial x^i. \tag{79}$$

This corresponds to re-aligning the old frames at each point of M to be tangent to the new coordinate curves. By 'lifting' structures from M into LM insight can be gained into their frame dependence. Since a local diffeomorphism $f: M \to M$ induces a mapping $f_*: TM \to TM$, any vector field X on M acquires a natural lift into LM. In the LM coordinates (x^μ, e_a^μ) the field $X = \varepsilon^\mu(x)\partial/\partial x^\mu$ on M is lifted into

$$\tilde{X} = X + e_a^\nu \frac{\partial \varepsilon^\mu(x)}{\partial x^\nu} \frac{\partial}{\partial e_a^\mu} \tag{80}$$

on LM under the one-parameter local diffeomorphism with representation $x^\mu \to x^\mu + \varepsilon^\mu(x)t + 0^\mu(t^2)$. The action of X on tensors on M can (Kobayashi and Nomizu 1969) be neatly formulated in terms of the action of $\tilde{X}$ in LM.

The geometry of M is defined in terms of a standard smooth choice of basis field in $T(LM)$. The $\mathrm{Gl}(4, R)$ fibre is spanned by 16 so-called vertical vector fields $E_s^{*r} \in vT(LM)$ which have a Lie algebra isomorphic to the Lie algebra of $\mathrm{Gl}(4, R)$. The remaining four dimensions are spanned by particular horizontal vector fields $B_k \in hT(LM)$. This decomposition is specified by a $\mathrm{Gl}(4, R)$ valued connection 1-form ω on LM such that in the vertical basis E_s^{*r} the 16 real forms $\omega^i{}_k$ obey

$$vX_u = \omega^i{}_k(X_u)E_i^{*k} \tag{81}$$

for any $X_u = vX_u + hX_u \in T_u(LM)$. In addition to these 16 real 1-forms the description of the basis in $T_u(LM)$ is completed by specifying four canonical 1-forms θ^a. Under the projection π_* induced from

$$\pi: LM \to M, (p, X_j) \to p \tag{82}$$

these are defined to satisfy

$$\pi_*(X_u) = \theta^j(X_u)X_j. \tag{83}$$

These forms and fields are chosen to satisfy the complete duality properties

$$\omega^i{}_k(E_s^{*r}) = \delta_s^i \delta_k^r \tag{84}$$

$$\omega^i{}_k(B_r) = 0 \tag{85}$$

$$\theta^i(B_k) = \delta_k^i \tag{86}$$

$$\theta^i(E_s^{*r}) = 0. \tag{87}$$

If we introduce the inverse vierbein E_μ^k satisfying

$$E_\mu^k e_i^\mu = \delta_i^k \tag{88}$$

and regard it as a function of the e_i^μ then in the specified coordinates of LM we may verify

$$\omega^k{}_j = E_\mu^k \, \mathrm{d}e_j^\mu + E_\mu^k \Gamma_{\alpha\nu}^\mu(x) e_j^\nu \, \mathrm{d}x^\alpha \tag{89}$$

$$\theta^i = E_\nu^i \, \mathrm{d}x^\nu \tag{90}$$

where $\Gamma_{\alpha\nu}^\mu$ are a set of 64 functions on M that specify a $\mathrm{Gl}(4, R)$ connection in these coordinates. In terms of the familiar covariant derivative on M

$$\nabla_{\partial/\partial x^\alpha}\left(\frac{\partial}{\partial x^\beta}\right)=\Gamma^\gamma_{\alpha\beta}\frac{\partial}{\partial x^\gamma}. \tag{91}$$

Now just as an arbitrary vector field X on M could be given a natural lift $\tilde{X}$ into LM one can give it a distinct 'horizontal lift' $\overset{\approx}{X}$ so that the lifted vector field lies in the horizontal subspace fixed by the connection in LM. The condition $\omega(\overset{\approx}{\partial/\partial x^\mu})=0$ gives in our working chart

$$\frac{\overset{\approx}{\partial}}{\partial x^\mu}=\frac{\partial}{\partial x^\mu}-\Gamma^\alpha_{\mu\beta}e^\beta_k\frac{\partial}{\partial e^\alpha_k}. \tag{92}$$

Thus there are two fundamentally distinct methods of differentiation with different geometrical interpretation on LM. The first requires a vector field associated with a one-parameter diffeomorphism on M for its specification, the second needs a linear connection on LM. Since the latter is associated with the geometry of M, particularly when we introduce a metric, we argue that it should have a unique correspondence with a dynamical formulation of gravity. The former is a tool that is independent of geometry and may be used in analysing any physical field configuration over M.

In terms of the standard basis on LM we can calculate

$$[E^{k*}_r, E^{k'*}_{r'}]=\delta^k_{r'}E^{k'*}_r-\delta^k_{r'}E^{k*}_r \tag{93}$$

$$[E^{*r}_s, B_k]=\delta^r_k B_s \tag{94}$$

and the system is closed if we introduce the functions $\overset{\approx}{T}{}^l_{ik}$, $\overset{\approx}{R}{}^r_{sik}$ on LM by

$$[B_i, B_k]=-\overset{\approx}{T}_{ik}{}^l B_l-\overset{\approx}{R}_{ik}{}^r{}_s E^{*s}_r. \tag{95}$$

These structure equations can be equivalently formulated in terms of the forms $\omega^i{}_j$ and θ^k

$$\overset{\approx}{R}{}^i{}_k=\tfrac{1}{2}\overset{\approx}{R}_{lm}{}^i{}_k\theta^l\wedge\theta^m=\mathrm{d}\omega^i{}_k+\omega^i{}_j\wedge\omega^j{}_k \tag{96}$$

$$\overset{\approx}{T}{}^i=\tfrac{1}{2}\overset{\approx}{T}_{lm}{}^i\theta^l\wedge\theta^m=\mathrm{d}\theta^i+\omega^i{}_k\wedge\theta^k \tag{97}$$

which are the curvature and torsion 2-forms describing the geometry. In a particular local gauge σ

$$\sigma: M\to LM \qquad (x^\mu)\to(x^\mu, e^\mu_a(x)) \tag{98}$$

we have written

$$e^a=\sigma*\theta^a \tag{99}$$

$$\omega^a{}_b=\sigma*\omega^a{}_b \tag{100}$$

$$R^a{}_b=\sigma*\overset{\approx}{R}{}^a{}_b \tag{101}$$

$$T^a=\sigma*\overset{\approx}{T}{}^a \tag{102}$$

as the forms that enter in our actions on M. Before this choice of local gauge section it is important to notice that the role of the vierbein (and its inverse) is a set of coordinates. After the gauge sectioning it defines a function on M. Since the pull back $\sigma^*:\Lambda^*(LM)\to\Lambda^*(M)$ commutes with d, the structure equations appear as the usual definition of R and T on M.

Up to this point our discussion has been in terms of an arbitrary connection in LM. If the manifold M is endowed with a metric g in TM we can construct g-orthonormal frames. The bundle of g-orthonormal frames OM is a sub-bundle of LM. If the metric has a Lorentz structure (i.e. eigenvalues $-1, 1, 1, 1$) then OM has the structure group SO(3, 1) which is contained in Gl(4, R). A linear connection in OM is called a metric connection since it controls the parallel transport of frames that stay in OM. Restricting Gl(4, R) to the proper Lorentz group one may compare the structural algebra (93), (94), (95) with the algebra that Hehl calls a modified Poincaré group. In our approach this algebra defines the structure of the orthonormal frame bundle of M.

It is most important to realise that a metric connection in OM induces naturally a Gl(4, R) connection on M. Thus given $\nabla_{X_a}X_b = \Gamma^c_{ab}X_c$ where $\{X_a\}$ is a g-orthonormal frame then

$$\nabla_{\partial/\partial x^\mu}\left(\frac{\partial}{\partial x^\nu}\right) = \nabla_{E^a_\mu X_a}(E^b_\nu X_b) = E^a_\mu \nabla_{X_a}(E^b_\nu X_b)$$
$$= E^a_\mu\{X_a(E^b_\nu)X_b + E^b_\nu \Gamma^c_{ab} X_c\} = \Gamma^\alpha_{\mu\nu}\,\partial/\partial x^\alpha. \tag{103}$$

We henceforth restrict to a metric compatible connection and work in OM with the proper Lorentz group as structure group. A dynamical theory of gravity will be expected to specify a connection in the Lie algebra of the Lorentz group together with a field $E^a_\mu(x)$ that relates the g-orthonormal coframes e^a to arbitrary coordinate coframes

$$e^a = E^a_\mu(x)\,\mathrm{d}x^\mu. \tag{104}$$

Since these are g-orthonormal the Lorentz metric follows as

$$g = -e^0 \otimes e^0 + \sum_{k=1}^{3} e^k \otimes e^k \tag{105}$$

and is preserved by the structure group of OM. Although we do not discuss fermions in this paper their existence in finite dimensional representations of SL(2, C) necessitates consideration of the group that covers SO(3, 1). It is for this reason that we enlarge our picture to the bundle of spinor frames over M and refer to our formulation as an SL(2, C) gauge theory of gravity. The existence of self-gravitating fermion fields is one of the strongest motivations we have for regarding SL(2, C) to be the appropriate structure group in the frame bundle.

Next let us examine $\mathscr{L}_X e^a$ and $\mathscr{L}_X \omega^a{}_b$ as 1-forms on M taking X to be an arbitrary vector field on M with orthonormal components $e^a(X) = X^a$. For this purpose it is useful to introduce the operator i_X where for any p-form α, $i_X : \Lambda^p \to \Lambda^{p-1}$

$$(i_X\alpha)(Y_1, Y_2, \ldots, Y_{p-1}) = p\alpha(X, Y_1, \ldots, Y_{p-1}) \qquad \forall Y_i. \tag{106}$$

We can then use the identity $\mathscr{L}_X = i_X\mathrm{d} + \mathrm{d}i_X$ on forms. Thus

$$\begin{aligned}\mathscr{L}_X e^a &= i_X\,\mathrm{d}e^a + \mathrm{d}i_X e^a\\ &= i_X(T^a - \omega^a{}_b \wedge e^b) + \mathrm{d}X^a\\ &= i_X T^a - (i_X\omega^a{}_b)\wedge e^b + \omega^a{}_b X^b + \mathrm{d}X^a\\ &= i_X T^a - i_X\omega^a{}_b \wedge e^b + DX^a\end{aligned} \tag{107}$$

but

$$i_X\omega^a{}_b = i_X(\omega_{c,}{}^a{}_b e^c) = \omega_{c,}{}^a{}_b X^c = \varepsilon^a{}_b$$

since the connection is metric compatible. Thus $(i_X\omega^a{}_b)e^b = \varepsilon^a{}_b e^b \equiv \delta_g e^b$ is the variation induced in e by a pure infinitesimal SO(3, 1) gauge rotation with a particular set of parameters $\varepsilon^a{}_b$. Thus we may write

$$\mathscr{L}_X e^a = i_X T^a + DX^a - \delta_g e^a. \tag{108}$$

Similarly

$$\begin{aligned}\mathscr{L}_X\omega^{ab} &= i_X\,\mathrm{d}\omega^{ab} + \mathrm{d}i_X\omega^{ab}\\ &= i_X[R^{ab} - \omega^{ac}\wedge\omega_c{}^b] + \mathrm{d}(\omega_{f,}{}^{ab}X^f)\\ &= i_X R^{ab} + D\varepsilon^{ab}\end{aligned} \tag{109}$$

where we have set $\varepsilon^{ab} = \omega_{f,}{}^{ab}X^f$ and $D\varepsilon^{ab} = \mathrm{d}\varepsilon^{ab} - \varepsilon^{ac}\omega_c{}^b - \varepsilon^b{}_f\omega^{af}$. Again we recognise $D\varepsilon^{ab} = -\delta_g\omega^{ab}$ as the variation induced in the connection form by the same infinitesimal SO(3, 1) gauge rotation.

At this point we can make contact with the transformations in Hehl (1979, 1980). The necessary correspondence is afforded by

$$\begin{gathered}\varepsilon^\mu = E^\mu_a\varepsilon^a \qquad \mathscr{L}_X e^a \equiv \delta e^a_\mu\,\mathrm{d}x^\mu \qquad \mathscr{L}_X\omega^a{}_b \equiv \delta\omega_{\mu,}{}^a{}_b\,\mathrm{d}x^\mu\\ T^a = \tfrac{1}{2}T_{\mu\nu}{}^a\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \qquad R^a{}_b = \tfrac{1}{2}R_{\mu\nu}{}^a{}_b\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \qquad D = \mathrm{d}x^\mu\nabla_\mu.\end{gathered} \tag{110}$$

(107) and (109) then become

$$\delta e^a_\mu = \nabla_\mu\varepsilon^a + \varepsilon^\nu T_{\nu\mu}{}^a - \varepsilon^a{}_b e^b_\mu \tag{111}$$

$$\delta\omega_\mu{}^a{}_b = \nabla_\mu\varepsilon^a{}_b + \varepsilon^\nu R_{\nu\mu}{}^a{}_b. \tag{112}$$

Apart from letter conventions these correspond to what Hehl (1979, 1980) terms pure 'modified Poincaré gauge translations'. If we add to the particular parameters $\varepsilon^a{}_b$ a set of six arbitrary parameters the above transformations may be interpreted as the component expressions for

$$[\mathscr{L}_X + \delta_{\mathrm{SO}(3,1)}]e^a \tag{113}$$

and

$$[\mathscr{L}_X + \delta_{\mathrm{SO}(3,1)}]\omega^a{}_b \tag{114}$$

where $\delta_{\mathrm{SO}(3,1)}$ denotes the combined infinitesimal SO(3, 1) gauge rotation. Thus these transformations have their origin in the behaviour of the fundamental structures e^a and $\omega^a{}_b$, associated in a certain gauge within LM, under the combined effect of an arbitrary diffeomorphism on M and a local Lorentz transformation.

To what extent do these transformations yield a covariance of a physical theory of gravity? To accommodate the principle of independence under coordinate labelling any field theory can be generated from a 4-form action density Λ. If the manifold over which it is integrated is closed, or if appropriate boundary conditions are imposed, then the action is invariant under arbitrary diffeomorphisms on M. This follows since

$$\mathscr{L}_X\int_M\Lambda = \int_M\mathscr{L}_X\Lambda = \int_M(i_X\,\mathrm{d}\Lambda + \mathrm{d}i_X\Lambda) = \int_M\mathrm{d}i_X\Lambda = \int_{\partial M}i_X\Lambda = 0 \tag{115}$$

where we use the fact that no 5-forms exist on M. Since this argument is valid for any 4-form action we do not believe $\mathscr{L}_X$ should be identified with a gauge structure for gravity in the same way as $\delta_{\mathrm{SO}(3,1)}$. Whether the action is invariant under this transformation, $\delta_{\mathrm{SO}(3,1)}\int_M\Lambda = 0$, depends on its construction of course. To construct an invariant action out of a linear connection in OM and the canonical forms (together with possible sections of bundles associated with LM) is in our view the simplest possible metric compatible formulation of gravity that generalises the original torsion-free formulation. It makes precise the notions of Cartan (1922) which are beautifully adapted to modern concepts in differential geometry. By comparison with the gauge theories of internal symmetries over M the theory is naturally regarded as a Lorentz group gauge theory of orthonormal space–time frames. The fact that we have recast the particular model of Baekler *et al* (1980) into this language re-enforces our opinion that the role of arbitrary diffeomorphisms on M via the transformations (113), (114) is simply a way of classifying e^a and $\omega^a{}_b$ consistently with the structure equations (96), (97) of the frame bundle.

4.

In this section we mention briefly a particular Cartan connection in the bundle of affine frames (Kobayashi and Nomizu 1969, Trautman 1973) over M that has often been used to interpret the linear connection in LM. In the language of gauge theory establishing a Cartan connection corresponds to gauging the Poincaré group by constructing a horizontal subspace in the bundle of g-orthonormal affine frames. An affine frame of M may be identified with a linear frame together with a point in the tangent space of M regarded as an affine space. This means that the vector $V_p \in T_pM$ with components $\{V_a\} \in R^4$ responds to the affine transformation

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

where $A \in \mathrm{Gl}(4, R)$, $b \in R^4$ by

$$V_a \to A_a^b V_b + b_b \tag{116}$$

and the affine group structure is obtained by multiplying the above matrices. One may visualise the affine transformation as linearly transforming the general linear frame and translating its origin in the tangent space to each point of the manifold M. Kobayashi and Nomizu (1969) show how the set of all affine frames may be turned into an affine frame bundle AM. Fixing a connection in the sub-bundle of g-orthonormal affine frames naturally induces an affine connection on the complete bundle. A Cartan connection in AM is a particular choice of affine connection that relates it to the linear connection in LM, the canonical 1-forms θ^a *and* the origin vector V. With this affine connection the curvature associated with the translational generators generated by b is related to the torsion in the linear frame bundle and the $\mathrm{Gl}(4, R)$ (or $\mathrm{SO}(3, 1)$) covariant derivatives of V.

If the structure group of the affine bundle is reduced to the linear group by fixing the origin of each affine frame firmly then, and only then, can we identify the translational connection with the canonical forms θ^a of LM and the translational curvature with the associated torsion. The affine symmetry is broken down to $\mathrm{Gl}(4, R)$ or $\mathrm{SO}(3, 1)$ and the affine bundle reduced to LM. This point appears to have been overlooked by recent proponents (Cho 1976, Hayashi 1977) of translation and Poincaré gauge description of gravity. To our knowledge there is no evidence in any theory of gravity of the fully gauged Poincaré symmetry group.

It is sometimes said (Schweizer *et al* 1980) that a knowledge of the forms (θ^a, $\omega_{ab} = -\omega_{ba}$) determines a Poincaré gauge connection. They also determine a connection in LM and in any bundle with SO(3, 1) as a subgroup of its structure group that can be reduced to LM. Consequently we believe that the status of the gauged Poincaré group is elusive in current models of gravitation and its theoretical role in interpreting (θ^a, $\omega^a{}_b$) be clearly distinguished from the role played by the Lorentz group as a local covariance group for gravitation. A more complete description of the affine frame bundle, generalised to accommodate supergravity, can be found in Tucker (1981).

5. Conclusion

In this paper we have computed a new solution to the model of gravitation introduced in Baekler *et al* (1980). The model has been discussed in Lorentz gauge covariant terms and the local symmetry exploited with the introduction of a convenient gauge in which the calculation was performed. However, Baekler *et al* (1980) have strongly advocated a description in terms of a 'modified Poincaré gauge group'. This description appears to be motivated by the apparent similarity of the Poincaré Lie algebra with the algebra of 'local P-transformations'. Unlike the Lie algebra of a structure group however this involves functions that vary with the geometry. In § 3 we have indicated how these relations arise as the structure equations of the linear frame bundle and should be regarded as providing a definition of the curvature and canonical forms. The local P-transformations of the 'modified Poincaré gauge potentials' are not canonical. However, we have identified them with a combined local Lorentz transformation generated by the action of the Lorentz structure group coupled with a Lie derivative action with respect to an arbitrary vector field on M. We prefer to dissociate the later transformation from any specific gauge interaction.

By formulating the classical laws of physics as relations between tensors and certain tensorial operators over a manifold, one can immediately generate laws relating their diffeomorphic images. Comparing tensors with their images under diffeomorphisms provides a succinct tool for the description of symmetries. Indeed we have exploited such techniques in § 2 where a spherically symmetric stationary geometry with torsion was defined. We also defined the Poincaré generators in terms of an infinitesimal isometry of the Minkowski metric with the aid of Lie derivatives. We stress, however, our contention that the tensorial formulation of physical laws and their behaviour under arbitrary diffeomorphisms should not be unique to any theory of gravitation.

Once the linear frame bundle LM has been adopted as the central concept in a gauge theory of gravity, one recognises its sections as gauge choices for the theory. An arbitrary change of gauge can be generated with an arbitrary vector field on LM and (at least locally) repositions the gauge section. If one uses as vector field the lift $\tilde{X}$ of a vector field X on M then LM has the characteristic that

$$\mathscr{L}_{\tilde{X}}\theta^a = 0 \qquad \forall \tilde{X}. \tag{117}$$

This follows simply in a local chart of LM from the expression given for $\tilde{X}$ in (80) and equation (90) for θ^a. However, given a connection, $\mathscr{L}_{\tilde{X}}\omega^a{}_b$ only vanishes for special fields $\tilde{X}$ and these define a class of affine symmetries on M.

For the purposes of constructing gravitational interactions with other fields the modified P-gauge approach claims to lead unambiguously to a minimal coupling procedure (see e.g. Hehl 1979, Lecture 3). In our approach all tensor and spinor fields (including the gauge fields of other interactions) are classified under the Lorentz group (Benn *et al* 1980). In principle there is no reason why a non-minimal coupling (for example between gravity and electromagnetism) should not occur that respects all local gauge covariances. An example of such an action 4-form would be $R^{ab} \wedge i_b(i_c F * R^c{}_a)$ where F is the U(1) invariant electromagnetic field 2-form. Similar interactions have been discussed in the literature (Prassana 1971, Horndeski 1976) and are fully consistent with our gauge approach.

Finally we remark on the important relation between space–time symmetries and conservation laws. The so-called 'covariant conservation laws' that follow from the invariance of the matter action under local Lorentz transformations and arbitrary diffeomorphisms on M are not really conservation laws, but rather consequences of the structure equations and the dynamical equations satisfied by non-gravitational fields. They are valid in arbitrary background geometries and place no restriction on the form of the background action should one wish to consider dynamical geometry. However, if the space admits symmetries generated by X on M such that $\mathscr{L}_X \mathbb{T} = 0, \mathscr{L}_X g = 0$, these identities lead to the existence of closed 3-forms (Benn 1981). For a Minkowski background with its ten-dimensional Poincaré symmetry these give rise to conserved angular momentum and momentum densities. Hehl and others have argued that since the conservation laws for angular momentum and momentum in special relativity result from global invariance under the Poincaré group this group must be fundamental to any gauge approach to gravity. Such theories, including Einstein's, treat the geometry as dynamical and provide field equations for the metric and connection. Minkowski space (with $T = 0$) is usually required to be one solution of the source-free equations. Since the conservation laws of special relativity follow from the global invariance of the Minkowski metric under a particular group of diffeomorphism on M, the Poincaré group, it seems unappealing to us to tie the group structure of a dynamical theory of gravitation to the properties of one particular solution.

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